Deutsch algorithm

$\mathbb{B}=\{0,1\}$

No.
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問題: An oracle $f:\mathbb{B}\longrightarrow\mathbb{B}$, is f constant or balanced?

神喻 (oracle) = blackbox (知道 I/O, 不知內部構造的運作)

f	$x=0$	1	type
f_1	0	0	constant
f_2	0	1	balanced
f_3	1	0	balanced
f_4	1	1	constant

Algorithm: $(H\otimes I)\,U_f\,(H\otimes H)$

說明:

(1) $U_f: |x,y\rangle \longrightarrow |x, y\oplus f(x)\rangle = \begin{cases} |x,y\rangle & f(x)=0 \\ |x,\bar{y}\rangle & f(x)=1 \end{cases}$

(2) $: |x\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} \rightarrow |x\rangle\otimes\frac{|0\oplus f(x)\rangle-|1\oplus f(x)\rangle}{\sqrt{2}} = \begin{cases} |x\rangle(|0\rangle-|1\rangle) & f(x)=0 \\ -|x\rangle(|0\rangle-|1\rangle) & f(x)=1 \end{cases}$

$= (-1)^{f(x)}|x\rangle(|0\rangle-|1\rangle)$

$: \frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} \rightarrow \begin{cases} \frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} = |+\rangle|-\rangle & f=f_1 \ \text{const} \\ \frac{|0\rangle-|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} = |-\rangle|-\rangle & f_2 \ \text{balanced} \\ \frac{-|0\rangle+|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} = -|-\rangle|-\rangle & f_3 \ \text{''} \\ \frac{-|0\rangle-|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} = -|+\rangle|-\rangle & f_4 \ \text{const} \end{cases}$

$|+\rangle\otimes|-\rangle$

(3) 備製

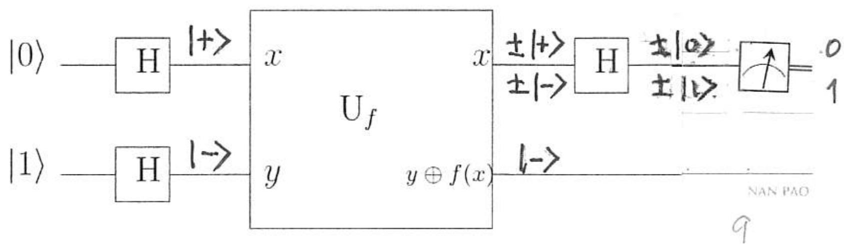
$|0\rangle\otimes|1\rangle \xrightarrow{H\otimes H} \frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}} = |+\rangle\otimes|-\rangle$

$\xrightarrow{U_f} \begin{cases} \pm|+\rangle|-\rangle & f:\text{const} \\ \pm|-\rangle|-\rangle & \text{balanced} \end{cases}$

$\xrightarrow{H\otimes I} \begin{cases} \pm|0\rangle|-\rangle & f:\text{const} \\ \pm|1\rangle|-\rangle & \text{else} \end{cases}$

$\xrightarrow{\text{測量}} \begin{cases} |0\rangle & f:\text{const} \\ |1\rangle & \text{else} \end{cases}$

(4) $\begin{cases} \text{call } U_f \text{ 一次} \\ \text{傳統上 call } f \text{ 2次.} \end{cases}$



Algorithm: Deutsch–Jozsa

$f : \mathbb{B}^n \to \mathbb{B}$

Inputs: (1) A black box U_f which performs the transformation $|x\rangle|y\rangle \to |x\rangle|y \oplus f(x)\rangle$, for $x \in \{0, \dots, 2^n - 1\}$ and $f(x) \in \{0, 1\}$. It is promised that $f(x)$ is either *constant* for all values of x, or else $f(x)$ is *balanced*, that is, equal to 1 for exactly half of all the possible x, and 0 for the other half.

Outputs: 0 if and only if f is constant.

Runtime: One evaluation of U_f. Always succeeds.

Procedure:

1. $|0\rangle^{\otimes n}|1\rangle$

2. $$\overset{H^{\otimes(n+1)}}{\to} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

3. $$\overset{U_f}{\to} \frac{1}{\sqrt{2^n}} \sum_x (-1)^{f(x)} |x\rangle|-\rangle$$

$$|x\rangle(|0\rangle - |1\rangle)$$
$$\to |x\rangle\left(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle\right)$$
$$= (-1)^{f(x)} |x\rangle \left(|0\rangle - |1\rangle\right)$$

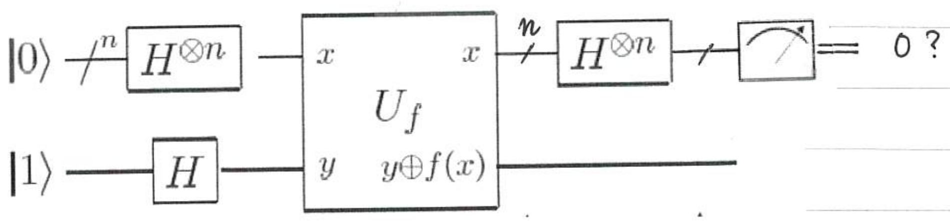
4. $$\overset{H^{\otimes n} \otimes I}{\to} \frac{1}{\sqrt{2^n}} \sum_x \left(H^{\otimes n}|x\rangle \otimes (-1)^{f(x)}|-\rangle\right)$$

$$= \frac{1}{2^n} \sum_x \left(\sum_z (-1)^{x \cdot z}|z\rangle \otimes (-1)^{f(x)}|-\rangle\right)$$

$$= \frac{1}{2^n} \sum_z \left[\left(\sum_x (-1)^{x \cdot z + f(x)}\right) |z\rangle \otimes |-\rangle\right].$$

5. $\xrightarrow{\text{measure on the first } n \text{ qubits}}$ z

$$\Pr(z = 0) = \frac{1}{2^n} \left|\sum_x (-1)^{f(x)}\right|^2 = \begin{cases} 1 & \text{if } f \text{ is constant,} \\ 0 & \text{if } f \text{ is balanced.} \end{cases}$$



$$|x\rangle\left(|0\rangle-|1\rangle\right) \xrightarrow{U_f} |x\rangle\left(|0\oplus f(x)\rangle-|1\oplus f(x)\rangle\right)$$
$$= (-1)^{f(x)}|x\rangle\left(|0\rangle-|1\rangle\right)$$

Example $n=2$

$$|001\rangle \xrightarrow{H^{\otimes 3}} \frac{1}{\sqrt{4}}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Case 1 $f(00)=f(01)=f(10)=f(11)=0$

$$\xrightarrow{U_f} \frac{1}{\sqrt{4}}\left(|00\rangle+|01\rangle+|10\rangle+|11\rangle\right)\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$= \frac{|0\rangle+|1\rangle}{\sqrt{2}}\,\frac{|0\rangle+|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$\xrightarrow{H^{\otimes 2}\otimes I} |00\rangle\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Case 2 $f(00)=f(01)=f(10)=f(11)=1$

$$\xrightarrow{U_f} \frac{1}{\sqrt{4}}\left(-|00\rangle-|01\rangle-|10\rangle-|11\rangle\right)\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$= -\frac{|0\rangle+|1\rangle}{\sqrt{2}}\,\frac{|0\rangle+|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$\xrightarrow{H^{\otimes 2}\otimes I} -|00\rangle\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Case 3 $f(00)=f(01)=0,\ f(10)=f(11)=1$

$$\xrightarrow{U_f} \frac{1}{\sqrt{4}}\left(|00\rangle+|01\rangle-|10\rangle-|11\rangle\right)\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$= \frac{|0\rangle-|1\rangle}{\sqrt{2}}\,\frac{|0\rangle+|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$\xrightarrow{H^{\otimes 2}\otimes I} |10\rangle\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Case 4 $f(00)=0,\ f(01)=1,\ f(10)=0,\ f(11)=1$

$$\xrightarrow{U_f} \frac{1}{\sqrt{4}}\left(|00\rangle-|01\rangle+|10\rangle-|11\rangle\right)\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$= \frac{|0\rangle+|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$
$$\xrightarrow{H^{\otimes 2}\otimes I} |10\rangle\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

Case 5 $\quad f(00)=0, \quad f(01)=f(10)=1, \quad f(11)=0$

$$\xrightarrow{U_f} \frac{1}{\sqrt{4}}\left(|00\rangle-|01\rangle-|10\rangle+|11\rangle\right)\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$= \frac{|0\rangle-|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}\,\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$\xrightarrow{U_f} |11\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$